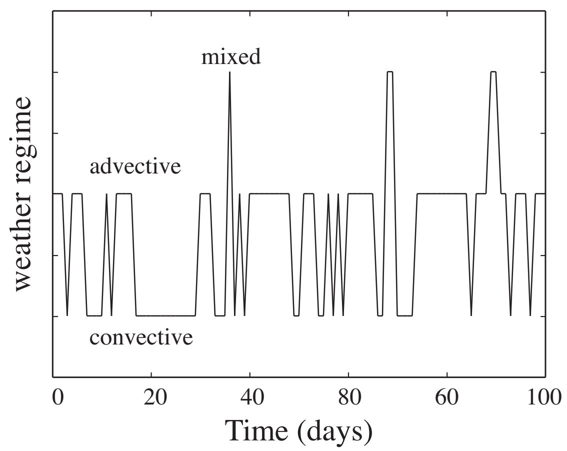
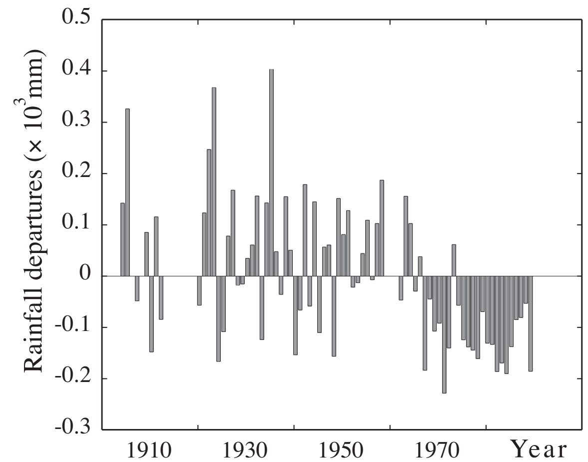
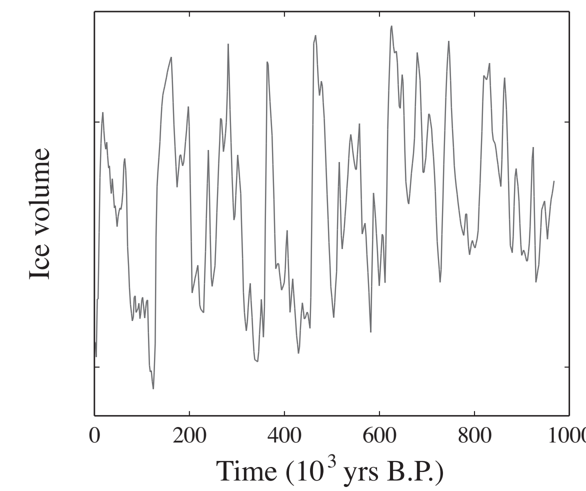
Climate response to externally induced time-dependent forcings : signatures and early warnings

Intrinsically-generated variability of the climatic system over a wide range of time scales, often manifested through the occurrence of large-scale transitions between different states.







Capturing and predicting these behaviors :

- Operational forecasting models, typically involving large numbers of variables and parameters.
- Concepts and methods of nonlinear dynamics, illustrated on low-order models.

Generic mechanisms at the basis of natural variability and of the transitions between states:

- Loss of stability of a certain “reference” state and bifurcation of new branches of solutions.
- Aperiodic behavior in the form of deterministic chaos.

Errors arising from model uncertainties. Stochastic parameterization of unresolved scales via error source terms modeled as Gaussian Markov noises.

Role of externally-induced forcings on the evolution of global climate: time-dependent "control" parameters ($\mathrm{CO_2}$ increase,...) interfering with the natural evolution laws. Need to disentangle natural variability from the effect of such externally-induced systematic biases when addressing the issue of climatic change.

Main thesis : Climatic change can be viewed as the response of a nonlinear dynamical system to time-dependent "control" parameters in the presence of noise,

$$\frac{dy_i}{dt} = \nu_i(\{y_j\}, \mu(t)) + R_i(t) \qquad\qquad i = 1, \cdots, n$$

where ν_i accounts for the principal kinetic and thermodynamic processes controlling the evolution of the variables y_i.

Goals :

- Can this type of forcing give rise to new transition phenomena between states or interfere with already existing ones.
- If so, can these transitions be anticipated by monitoring suitable "forerunner" observables.

Our strategy : take advantage of the reduction of the multivariate dynamics in the vicinity of certain kinds of transitions into a low-order one described by a universal *normal form* featuring a limited number of variables, the *order parameters* :

$$\frac{dx_i}{dt} = f_i(\{x_j\}, \lambda(t)) + F_i(t) \qquad i = 1, \cdots, m \quad m << n$$

where x, λ, F are combinations of y, μ and R.

Focus on the frequently encountered case of transitions between steady-state solutions (glacial to interglacial climate, zonal to blocked circulation, etc).

Normal form equations reduce then to a single equation for a single relevant order parameter

$$\frac{dx}{dt} = f(\{x\}, \lambda(t)) + F(t)$$

such that for λ constant and in absence of noise there exists a critical value λ_c where bifurcation of new branches of solutions is taking place.

Adopt parameter variability in the form of a ramp :

- $\lambda(t) = \lambda_0 + \epsilon t$ with $\lambda_0 < \lambda_c$, $|\epsilon/\lambda_0| << 1$ "forward case"
- $\lambda(t) = \lambda_0 - \epsilon t$ with $\lambda_0 > \lambda_c$, $|\epsilon/\lambda_0| << 1$ "backward case"

Two realizations of this setting:

- Supercritical pitchfork bifurcation.
- Limit point bifurcation.

Outline

1. Pitchfork bifurcation
   A. Noise-free system
   B. Effect of noise

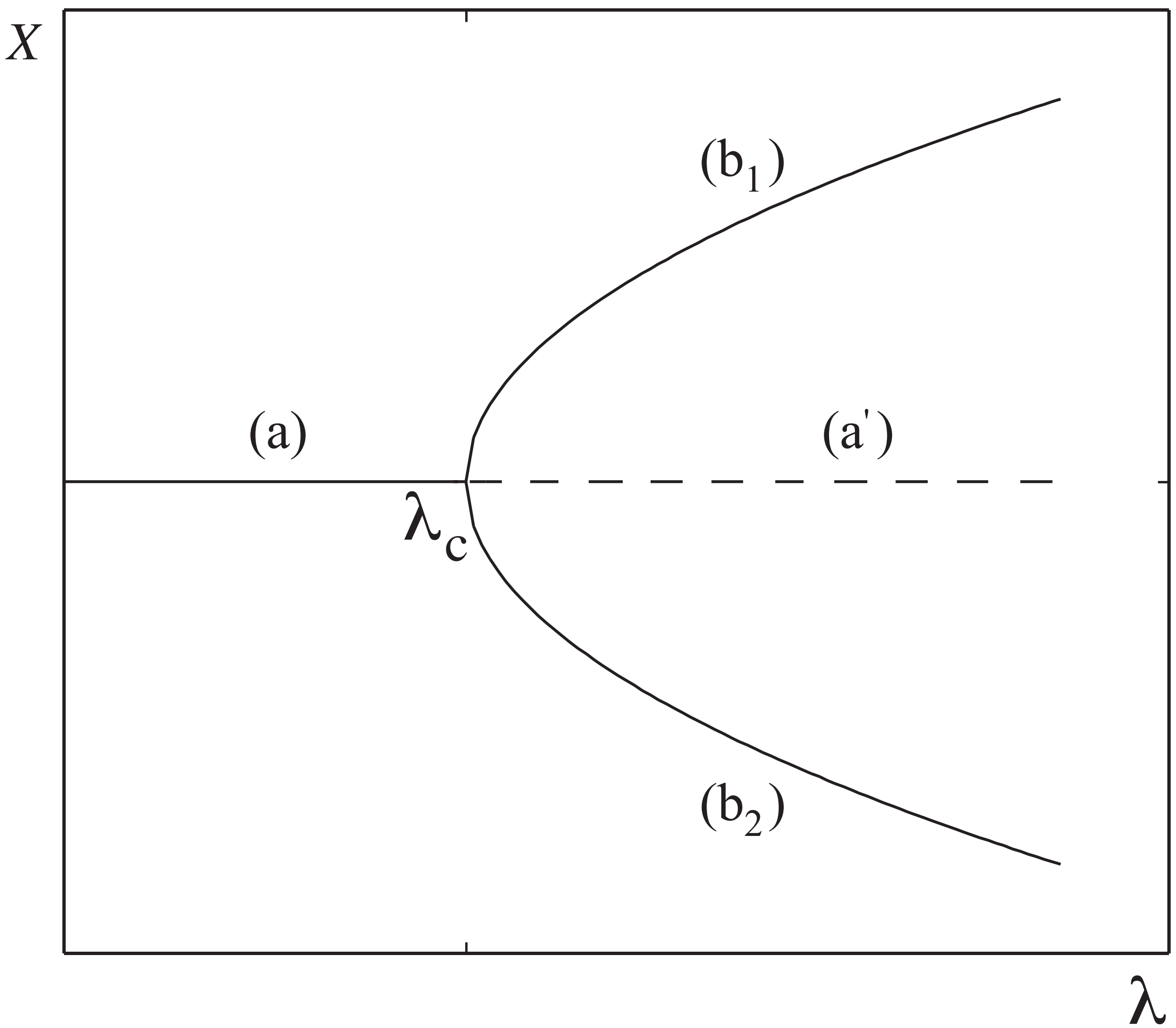
2. Limit point bifurcation
   A. Noise-free system
   B. Effect of noise

3. Illustration on a global energy balance model

4. Conclusions

1. Pitchfork bifurcation

A. Noise-free system



Normal form

$$\frac{dx}{dt} = \lambda(t)x - x^3$$

Fixed λ

$\lambda < 0$ unique stable state $x = 0$

$\lambda > 0$ $x = 0$ becomes unstable
$x_{\pm} = \pm\lambda^{1/2}$ are new stable states

$\lambda = 0$ bifurcation point

Time dependent λ

- Forward case $\lambda(t) = \lambda_0 + \epsilon t$, $\lambda_0 < 0$

  Exact solution

$$x(t) = \frac{x_0 \exp\{\lambda_0 t + \epsilon \frac{t^2}{2}\}}{\left[1 - i x_0^2 \sqrt{\frac{\pi}{\epsilon}} \exp(-\frac{\lambda_0^2}{\epsilon})(\mathrm{Erfi}\,\frac{\lambda_0 + \epsilon t}{\epsilon^{1/2}} - \mathrm{Erfi}\,\frac{\lambda_0}{\epsilon^{1/2}})\right]^{1/2}}$$

  Linearized version, suitable for short-time behavior

$$x(t) \approx x_0 \exp\left(\lambda_0 t + \epsilon \frac{t^2}{2}\right)$$

  Main effect : Switching to the non-trivial solution is delayed by a time

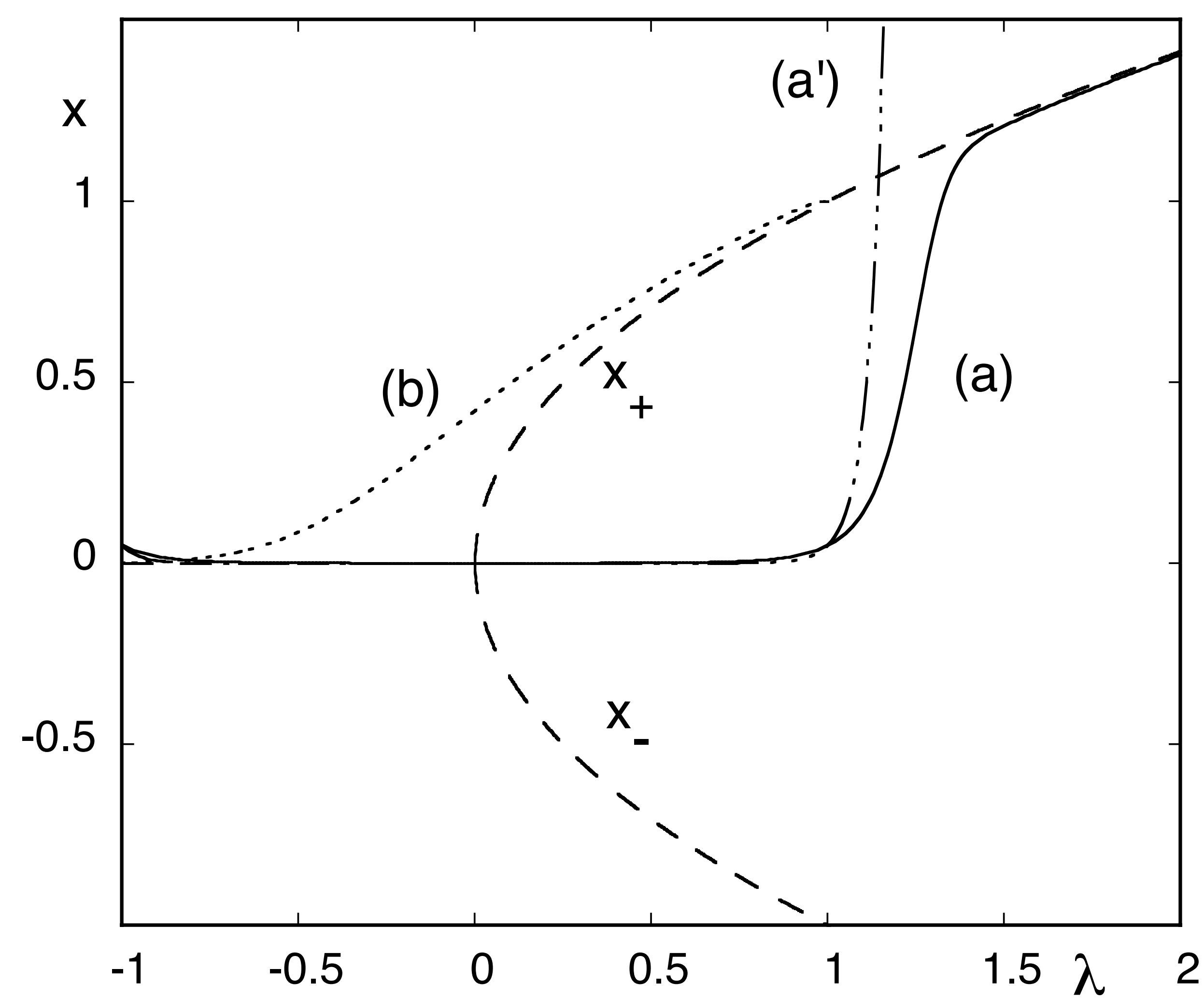
$$t_d = \frac{2|\lambda_0|}{\epsilon}$$

- Backward case $\lambda(t) = \lambda_0 - \epsilon t$, $\lambda_0 > 0$

Exact solution

$$x(t) = \frac{x_0 \exp\{\lambda_0 t - \epsilon \frac{t^2}{2}\}}{\left[1 + x_0^2 \sqrt{\frac{\pi}{\epsilon}} \exp(\frac{\lambda_0^2}{\epsilon})(-\mathrm{Erf}\frac{\lambda_0 - \epsilon t}{\epsilon^{1/2}} + \mathrm{Erf}\frac{\lambda_0}{\epsilon^{1/2}})\right]^{1/2}}$$

Main effect : Transition to the trivial branch $x = 0$ starting from state $x_+(0) = \lambda_0^{1/2}$ is likewise delayed.



B. Effect of noise

$$\frac{dx}{dt} = \lambda(t)x - x^3 + F(t)$$

$F(t)$ Gaussian white noise of strength q^2

Associated Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(\lambda(t)x - x^3)P + \frac{q^2}{2}\frac{\partial^2 P}{\partial x^2}$$

Average $< x >$ coupled to the variance $< \delta x^2 >$, $< \delta x^2 >$ coupled to $< \delta x^3 >$ and $< \delta x^4 >$, etc...

Moments

- Exact expressions in the adiabatic approximation :

$$
\begin{aligned}
< \delta x^2 >_t &= \frac{\lambda(t)}{2}\,\frac{K_{1/4}(\alpha) - K_{3/4}(\alpha)}{K_{1/4}(\alpha)} \qquad \text{for} \quad \lambda < 0 \\
< \delta x^2 >_t &= \frac{\lambda(t)}{2}\,\frac{K_{1/4}(\alpha) + K_{3/4}(\alpha) + \pi\sqrt{2}(I_{1/4}(\alpha) + I_{3/4}(\alpha))}{K_{1/4}(\alpha) + \pi\sqrt{2} I_{1/4}(\alpha)} \text{for} \quad \lambda > 0 \\
< \delta x^4 >_t &= \frac{q^2}{2}\,\frac{(\alpha + \frac{1}{8})K_{1/4}(\alpha) - \alpha K_{3/4}(\alpha)}{K_{1/4}(\alpha)} \qquad \text{for} \quad \lambda < 0 \\
< \delta x^4 >_t &= 2q^2\,\frac{\frac{1}{4}K_{1/4}(\alpha) + \pi\sqrt{2}[\alpha(I_{-3/4}(\alpha) + I_{-1/4}(\alpha)) + (\alpha + \frac{1}{4})I_{1/4}(\alpha) + \alpha I_{3/4}(\alpha)]}{K_{1/4}(\alpha) + \pi\sqrt{2} I_{1/4}(\alpha)} \text{for} \quad \lambda > 0
\end{aligned}
$$

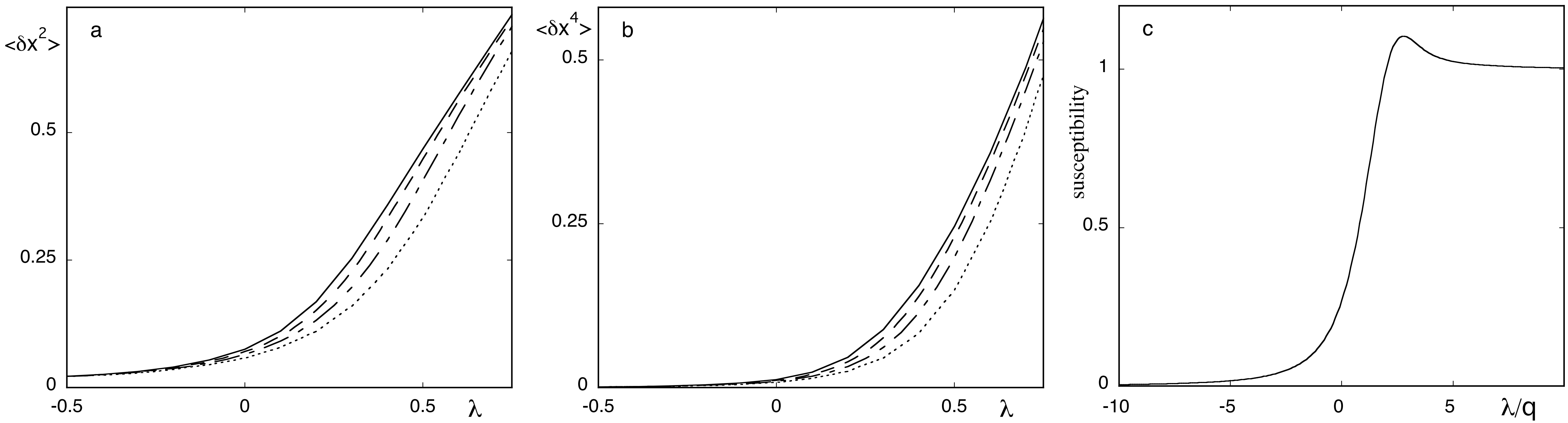
  where K and I are modified Bessel functions of fractional order, $I_{-\nu} = I_\nu + 2/\pi \sin(\pi\nu) K_\nu$ and α stands for $\lambda^2(t)/(4q^2)$.

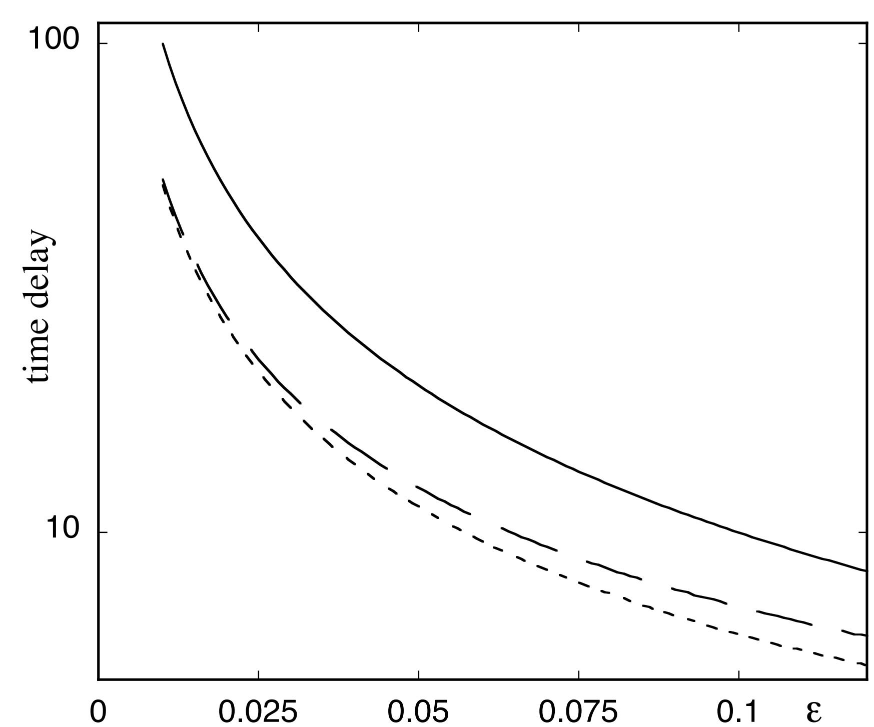
- Numerical integration of the Fokker-Planck equation for given noise strength and for different ϵ's.

Growth of $2^{\rm nd}$ and $4^{\rm th}$ order variances in the forward scenario: early warning.

Maximum of susceptibility, $\chi_\lambda = \frac{\partial < \delta x^2 >}{\partial \lambda}$.

Delays tend to be reduced by the noise.





Entropies

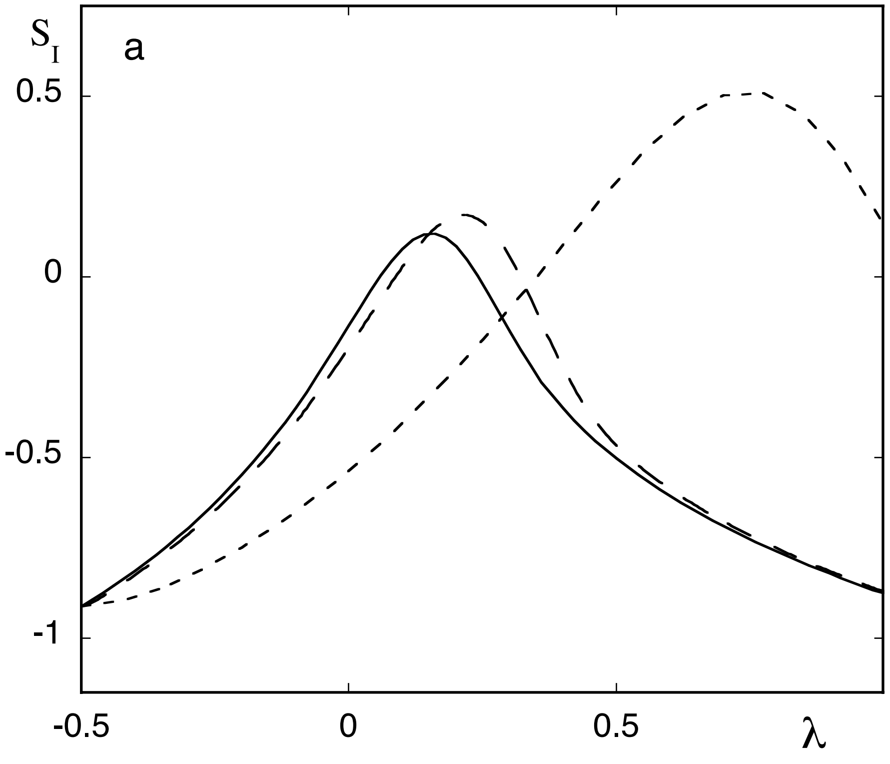
Information entropy

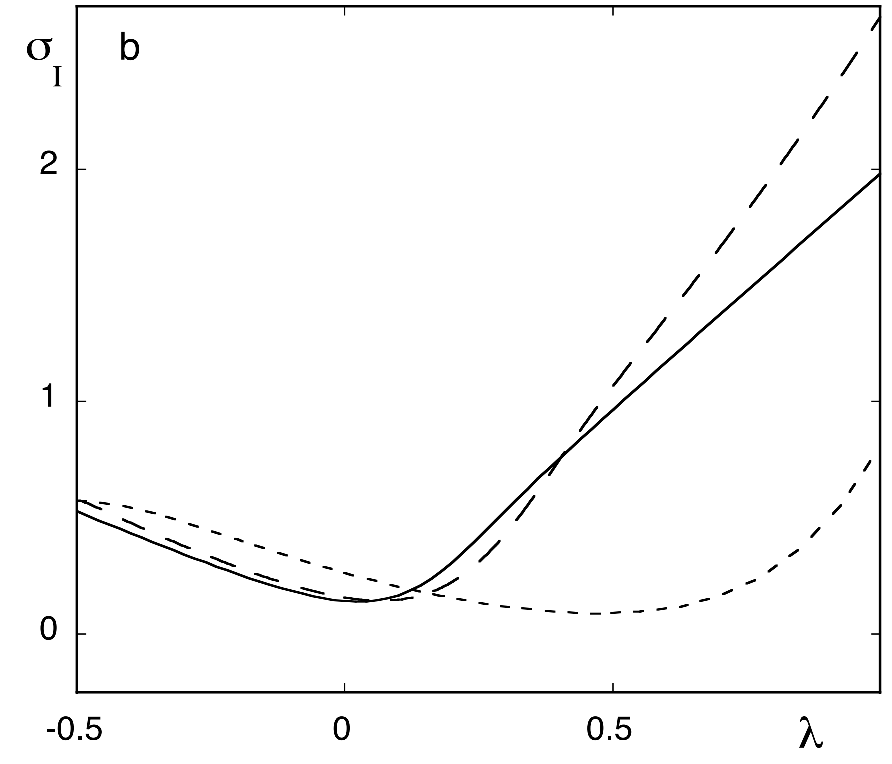
$$S_I(t) = -\int dx P(x,t) \ln P(x,t)$$

and information entropy production

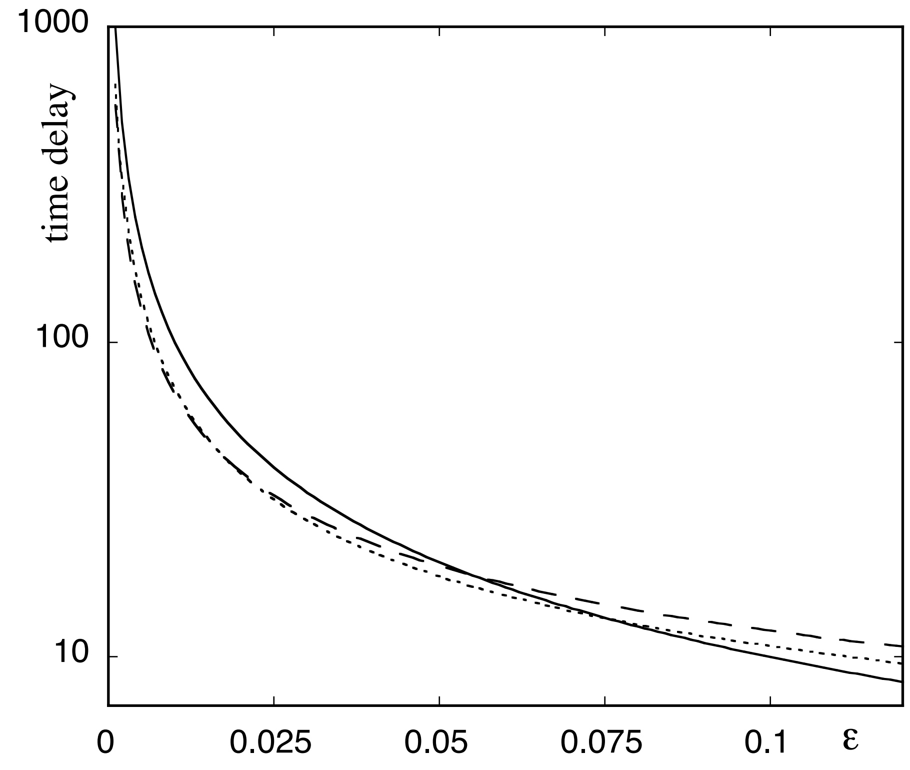
$$\sigma_I = \frac{q^2}{2} \int dx \frac{1}{P} (\frac{\partial P}{\partial x})^2 > 0$$

as global indicators
Extrema of both S_I and σ_I well beyond the transition point $\lambda = 0$, indicating delays associated to the reshuffling of the probability mass as the system gradually enters in the two-state region.

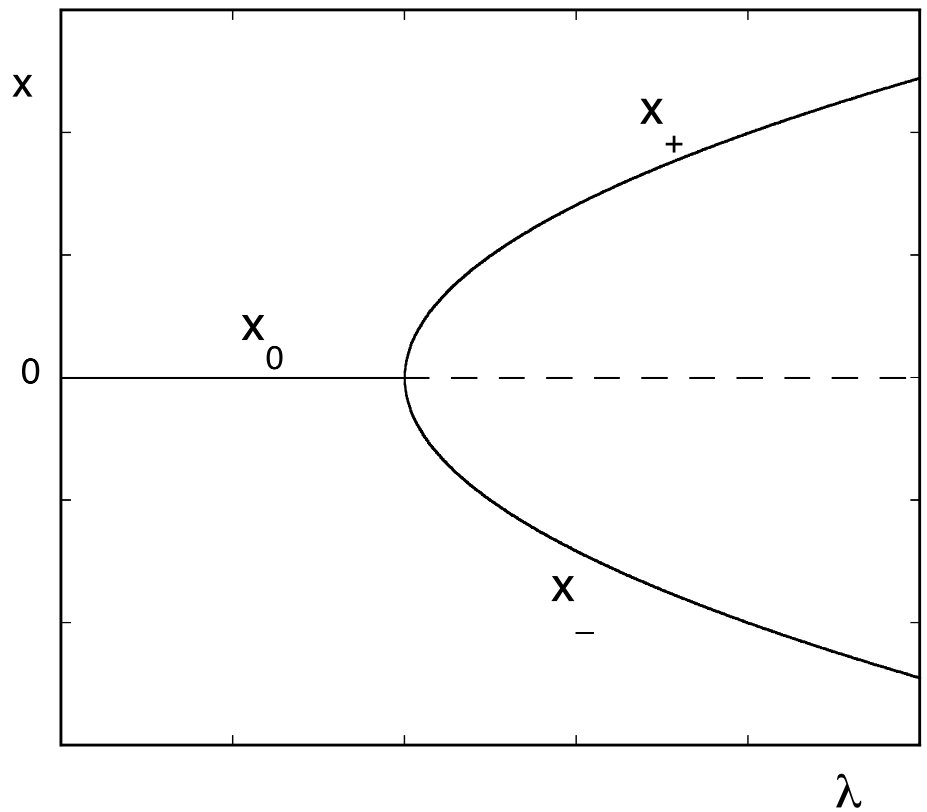




Cross-over of deterministic and stochastic delays.



Further indicators of global behavior : frozen states



On a long time scale, transitions are taking place between the "adiabatic" stable branches $x_{\pm} = \pm (\lambda_0 + \epsilon t)^{1/2}$ across the "barrier" associated to the presence of the intermediate unstable state $x = 0$.

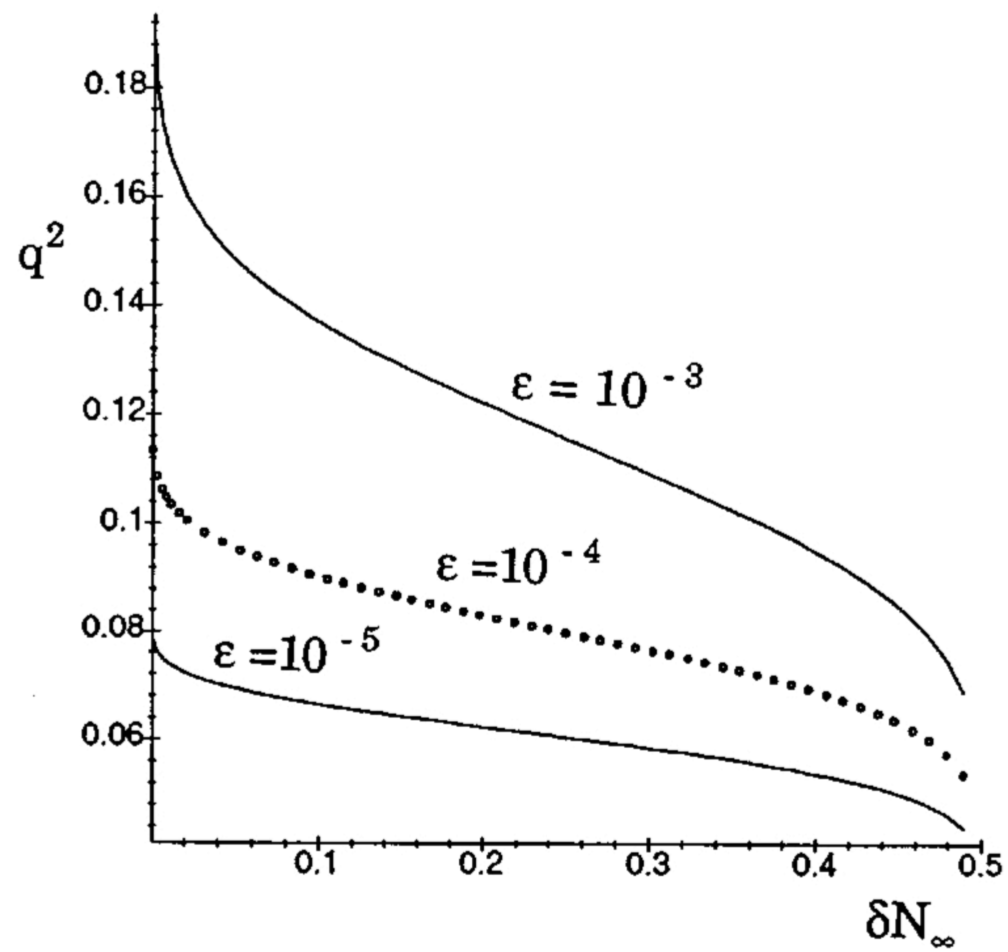
The characteristic time of these transitions is given by a generalization of the classical Kramers expression for the rate of an activated process :

$$\tau^{-1}(t) = \frac{\sqrt{2}}{\pi} (\lambda_0 + \epsilon t) \exp \left[-\frac{(\lambda_0 + \epsilon t)^2}{2q^2} \right]$$

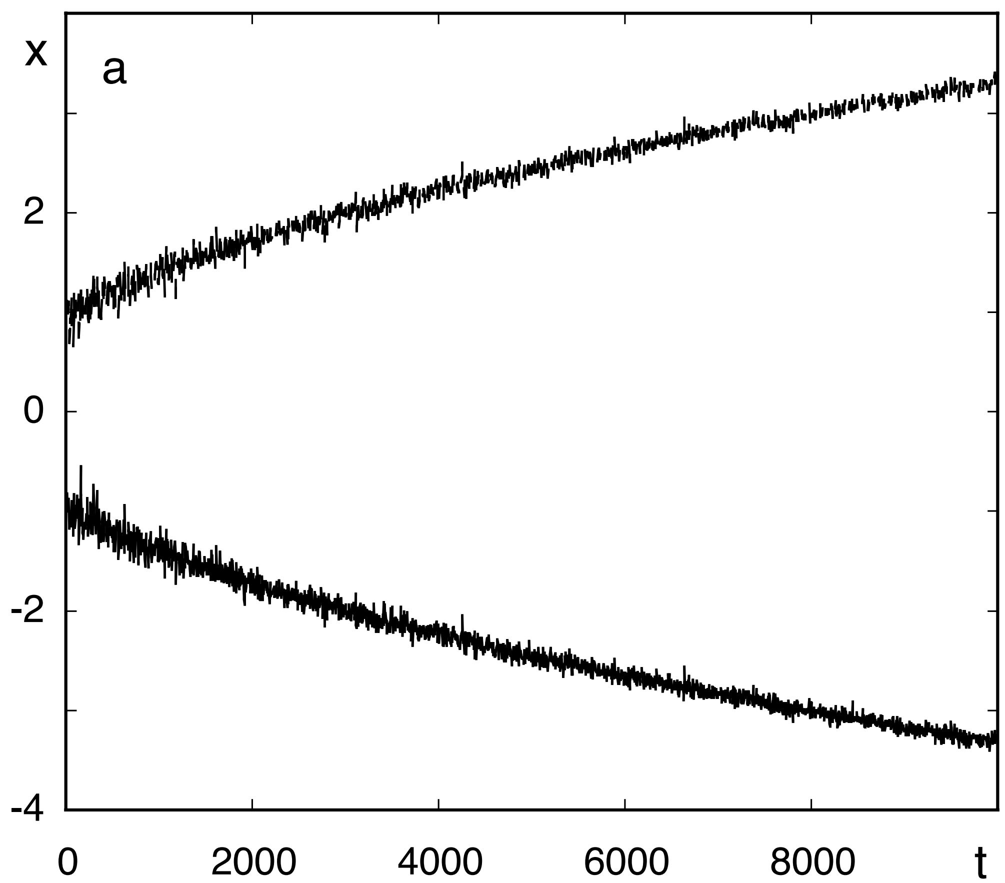
It follows that there exists a finite fraction of initial conditions in the quasi-attraction basin of $x_+(t)$ (or of $x_-(t)$) that will never cross the barrier, given by the expression

$$\delta N_{+,\infty} = N_{+,\infty} - \frac{1}{2} = \left(N_+(0) - \frac{1}{2}\right) \exp\left[-\frac{\sqrt{2}}{\pi}\frac{q^2}{\epsilon} \exp\left(-\frac{\lambda_0^2}{2q^2}\right)\right]$$

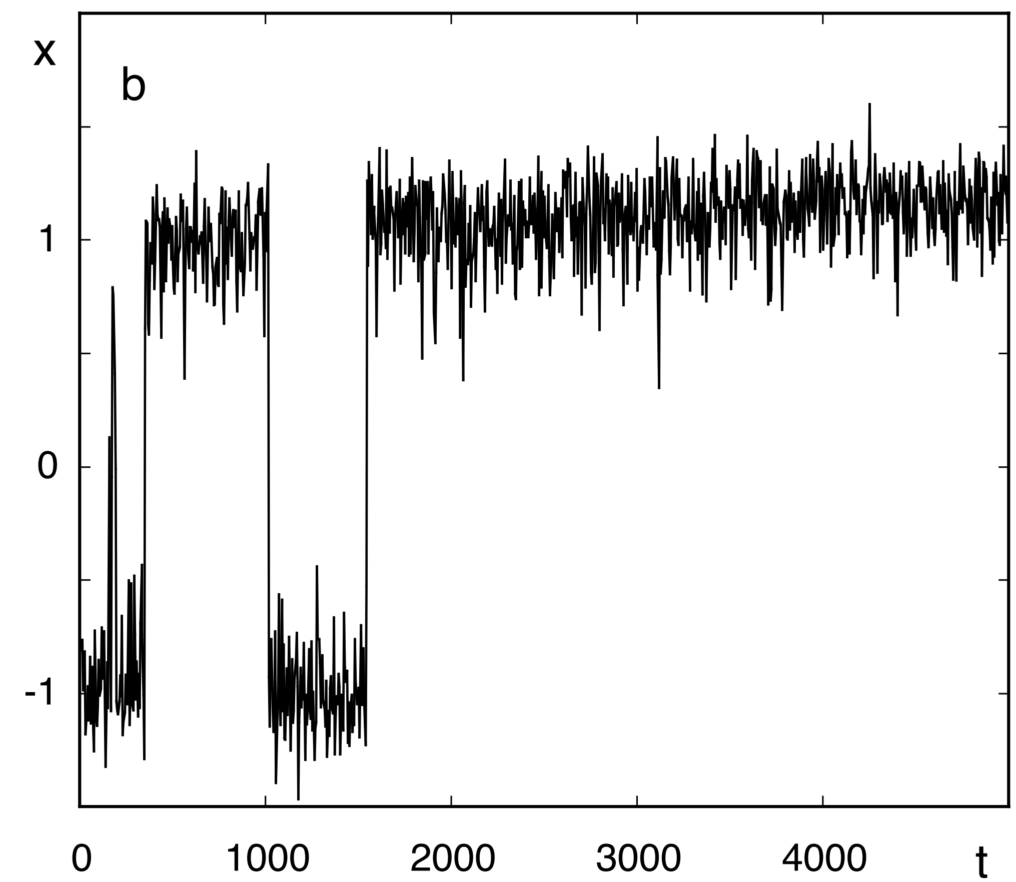
This faction depends very sensitively on ϵ and q^2.



Stochastic simulations (breakdown of ergodicity)



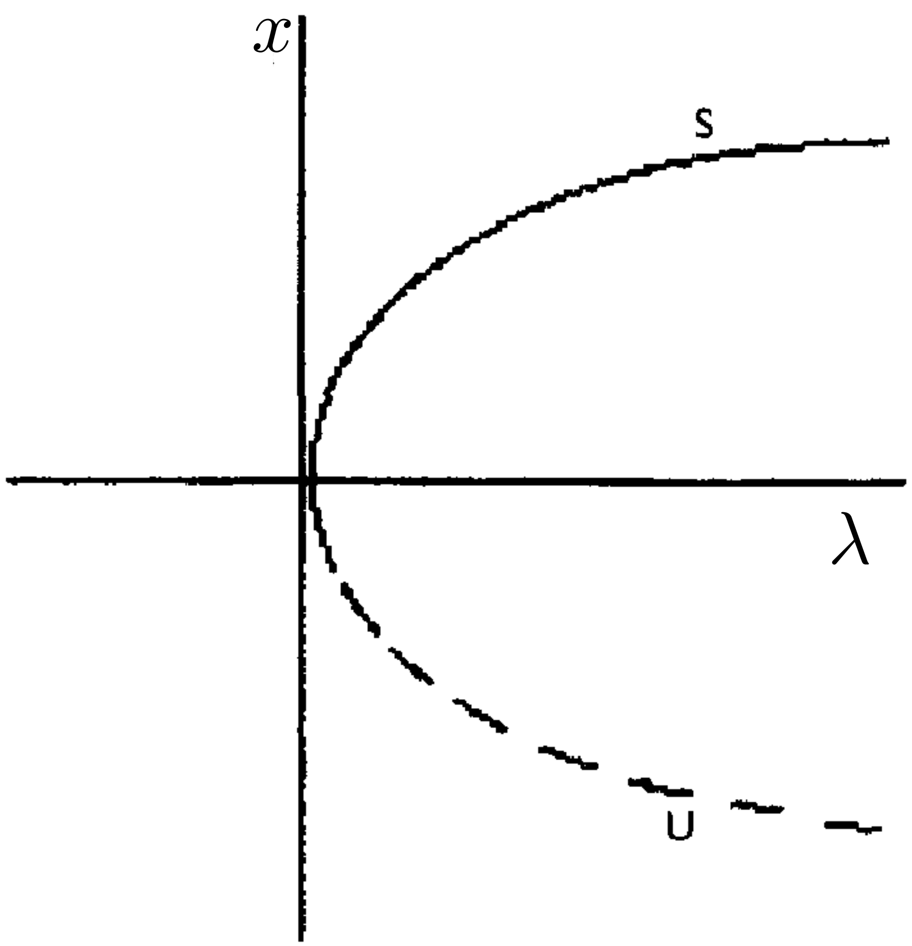
$\varepsilon = 10^{-3}$, $q^2 = 0.08$



$\varepsilon = 10^{-4}$, $q^2 = 0.12$

2. Limit point bifurcation

A. Noise-free system



Normal form

$$\frac{dx}{dt} = \lambda(t) - x^2$$

Fixed λ

$\lambda < 0$ no steady state, trajectories diverge to $-\infty$

$\lambda > 0$ $x_+ = \lambda^{1/2}$ is a stable steady state
$x_- = -\lambda^{1/2}$ is an unstable steady state

$\lambda = 0$ bifurcation point

Time dependent λ

Exact solution

$$x(t) = \pm\epsilon^{1/3} \frac{A'i(\frac{\lambda_0 \pm \epsilon t}{\epsilon^{2/3}}) + CB'i(\frac{\lambda_0 \pm \epsilon t}{\epsilon^{2/3}})}{Ai(\frac{\lambda_0 \pm \epsilon t}{\epsilon^{2/3}}) + CBi(\frac{\lambda_0 \pm \epsilon t}{\epsilon^{2/3}})}$$

where $\pm$ refer to the forward and backward cases respectively and C is determined by the initial condition

$$C = \frac{\pm\epsilon^{1/3} A'i(\frac{\lambda_0}{\epsilon^{2/3}}) - x_0 Ai(\frac{\lambda_0}{\epsilon^{2/3}})}{Bi(\frac{\lambda_0}{\epsilon^{2/3}}) x_0 \mp \epsilon^{1/3} B'i(\frac{\lambda_0}{\epsilon^{2/3}})}$$

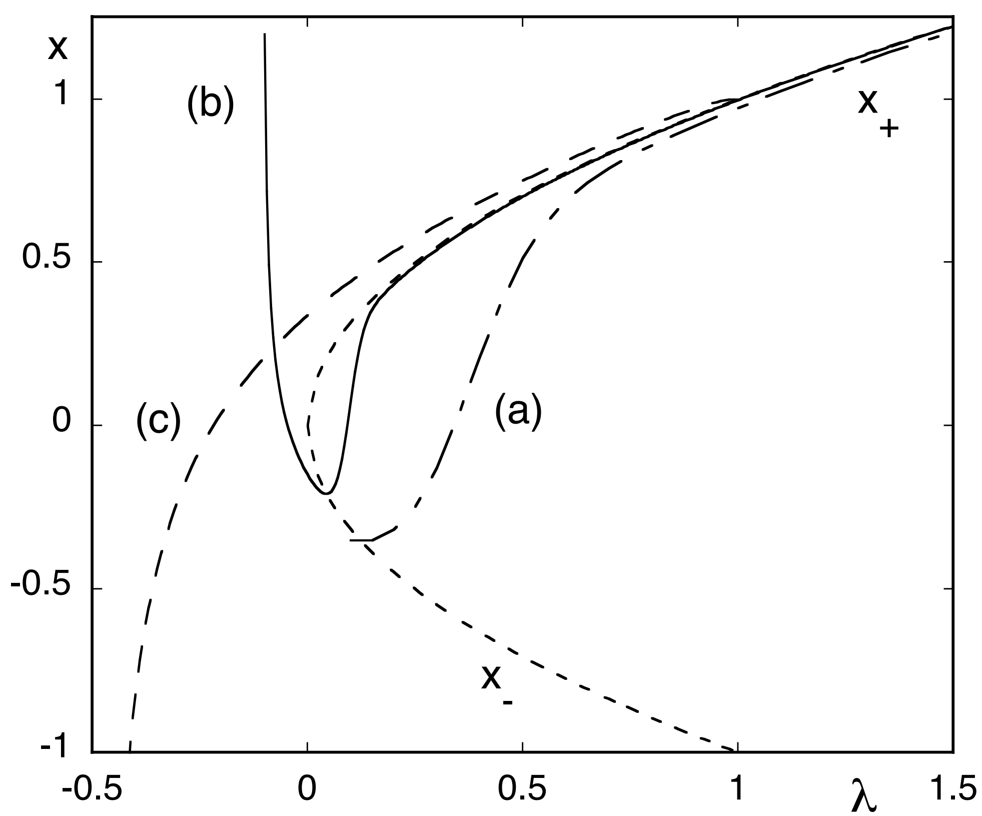
Ai, Bi are the Airy functions.

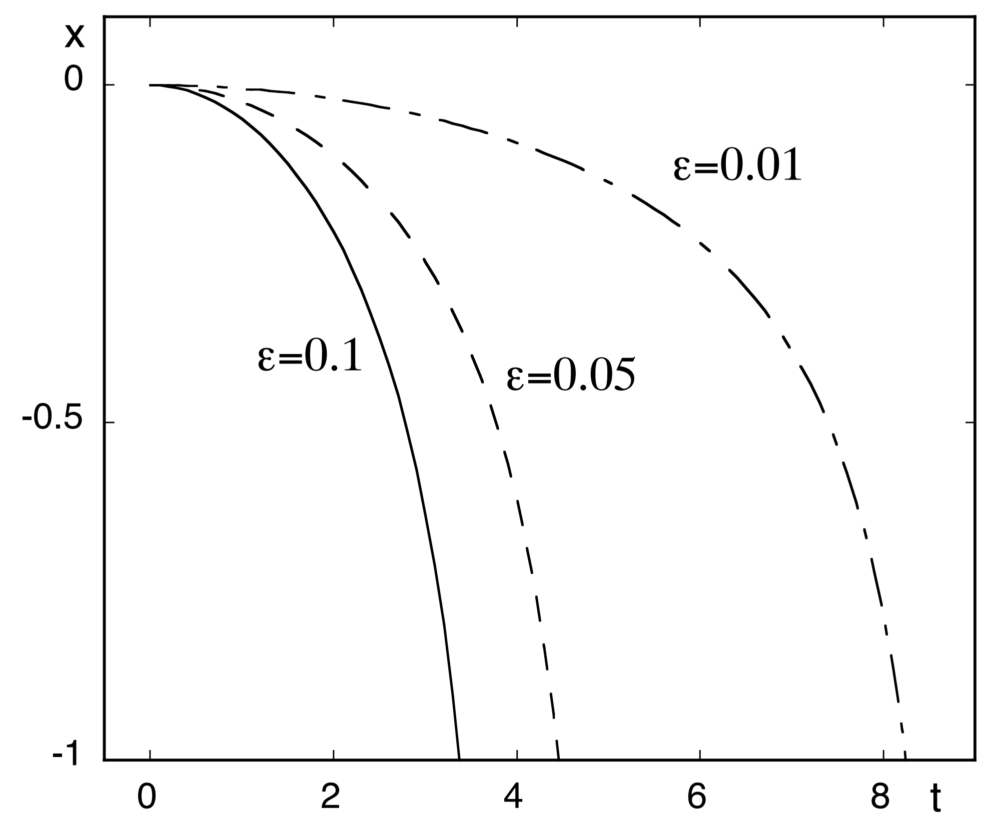
Main effects :

- Stabilization of a wide class of states that would otherwise diverge to $-\infty$, as long as initial conditions satisfy the inequality (forward case)

$$x_0 > \epsilon^{1/3} \frac{A'i(\frac{\lambda_0}{\epsilon^{2/3}})}{Ai(\frac{\lambda_0}{\epsilon^{2/3}})}$$

- Early warning in the form of slowing down (backward case).





B. Effect of noise

$$\frac{dx}{dt} = \lambda(t) - x^2 + F(t)$$

$F(t)$ Gaussian white noise of strength q^2

Associated Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(\lambda(t) - x^2)P + \frac{q^2}{2}\frac{\partial^2 P}{\partial x^2}$$

Average $< x >$ coupled to the variance $< \delta x^2 >$, $< \delta x^2 >$ coupled to $< \delta x^3 >$, etc. . .

Analytic evaluation in the linearized regime provides a first understanding of the behaviour of the fluctuations

Starting point :

$$\frac{d < \delta x^2 >}{dt} = -4\overline{x}(t) < \delta x^2 > + q^2$$

where $\overline{x}(t)$ satisfies the normal form equation in the absence of noise

Solution :

$$< \delta x^2 >_t = q^2 \frac{1}{Y^4(\frac{\lambda_0 \pm \epsilon t}{\epsilon^{2/3}})} \int_0^t dt_1 Y^4(\frac{\lambda_0 \pm \epsilon t_1}{\epsilon^{2/3}})$$

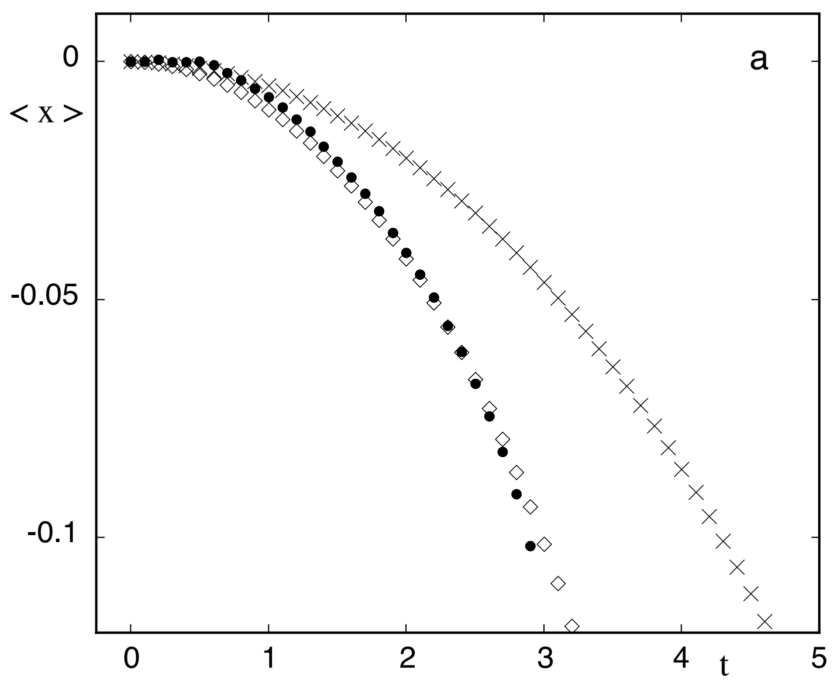
where Y is a linear combination of Airy functions.

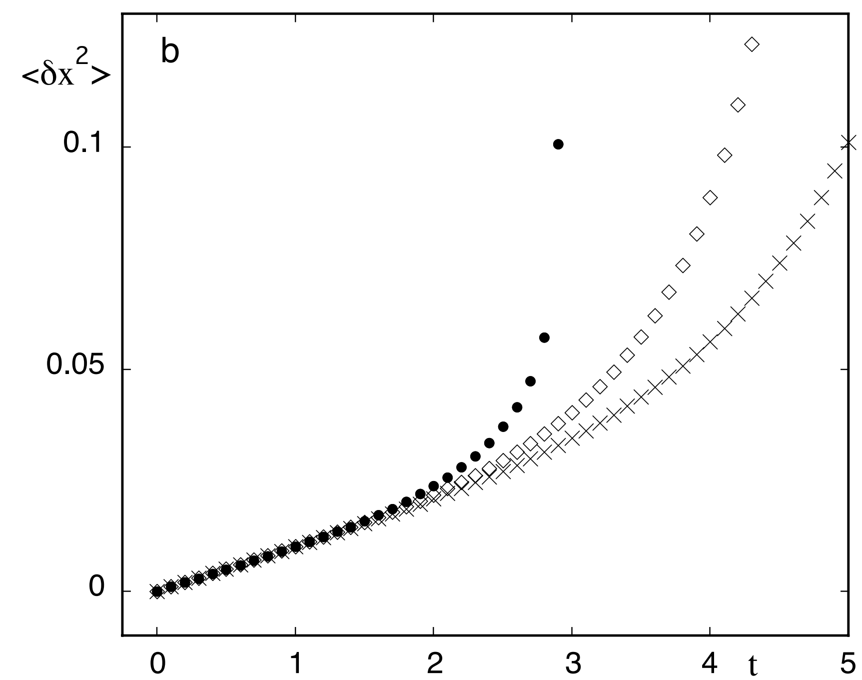
Diffusion-like behavior in the vicinity of the limit point (short times)

$$< \delta x^2 >_t \approx q^2 t$$

Slowing down followed by explosive behavior (backward scenario) : early warning.

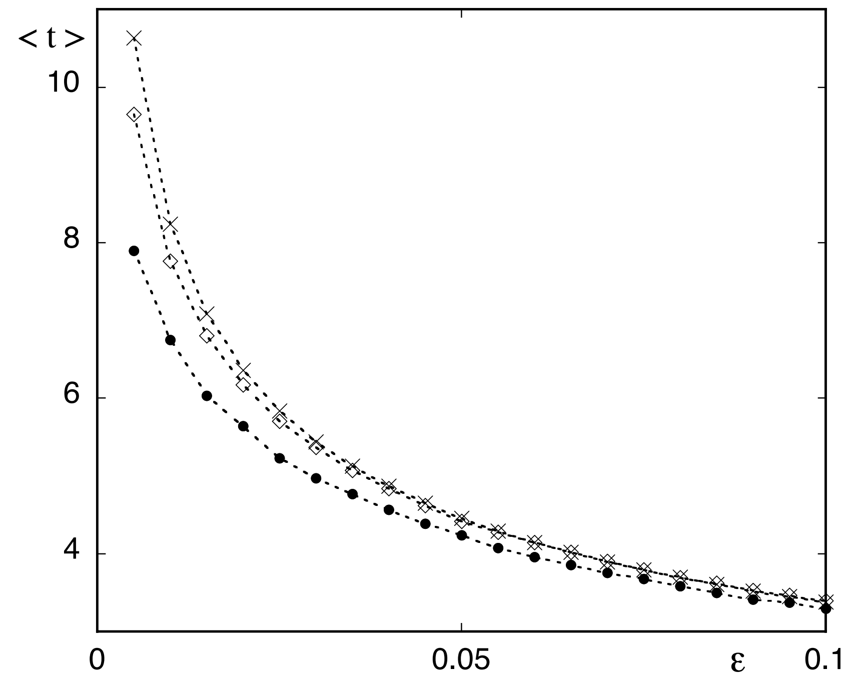
Coupling with fluctuations accelerates explosive behavior.



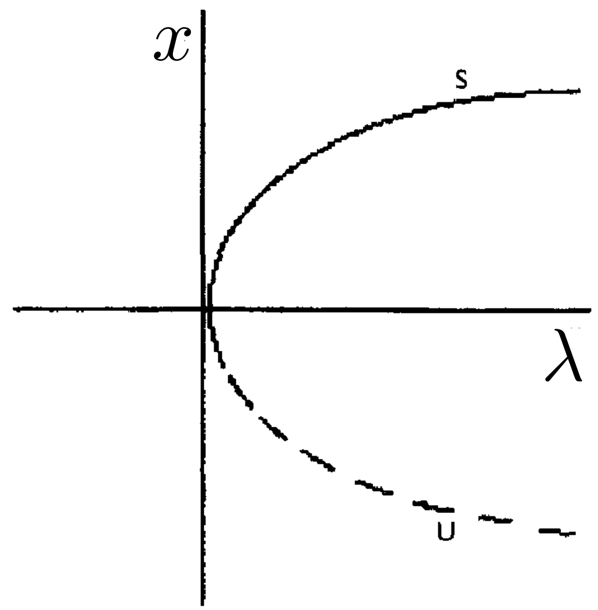


Fluctuation-driven evolutions.

Mean waiting times for crossing a prescribed threshold decrease with ϵ and q^2.



Frozen states



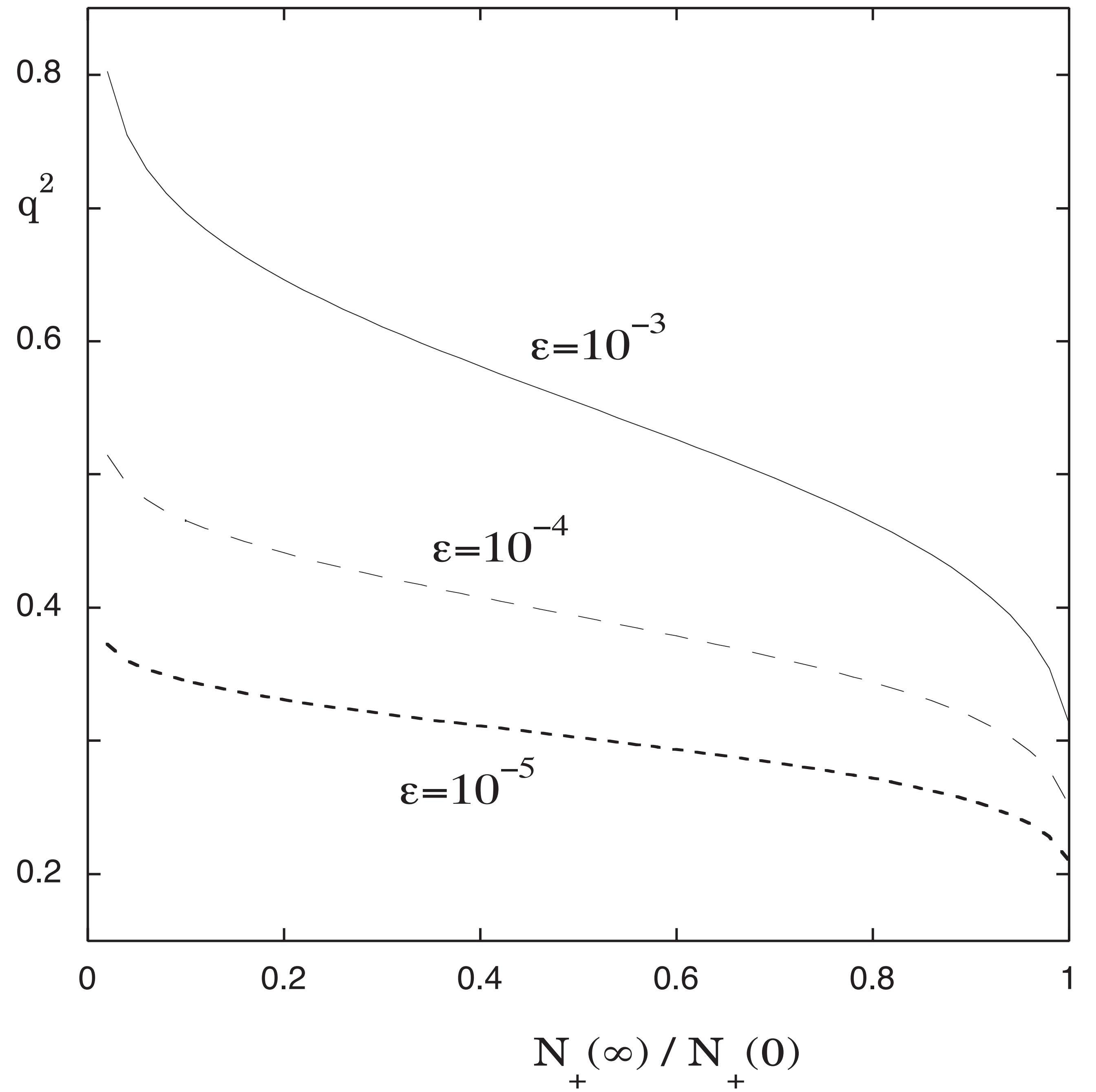
Constant λ: all trajectories eventually escape to $-\infty$ following fluctuation-induced jumps across the barrier separating x_+ from x_-, with a characteristic time which is an increasing function of λ and is given again by a Kramers-type expression

$$\tau = \frac{\pi}{2\lambda^{1/2}} \exp\left[\frac{8\lambda^{3/2}}{3q^2}\right]$$

Time-dependent λ, forward scenario: there exists a finite fraction of initial conditions in the quasi-attraction basin of $x_+(t)$ that will never escape, given by

$$N_{+,\infty} = N_+(0) \exp\left[-\frac{q^2}{2\pi\epsilon} \exp\left(-\frac{8\lambda^{3/2}}{3q^2}\right)\right]$$

This fraction depends again very sensitively on ϵ and q^2



3. Illustration on a global energy balance model

Energy balance equation for the globally averaged temperature T

$$C\frac{dT}{dt} = Q\left(1 - a(T)\right) - I(T)$$

C heat capacity

Q solar constant

a Albedo

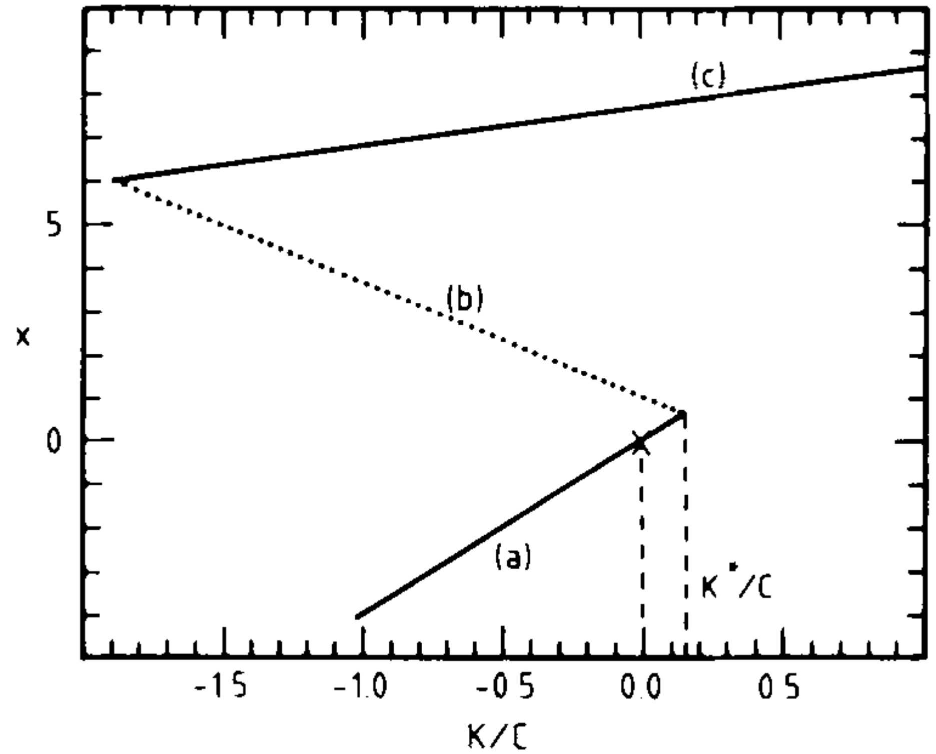
I Infrared cooling

Minimal model :

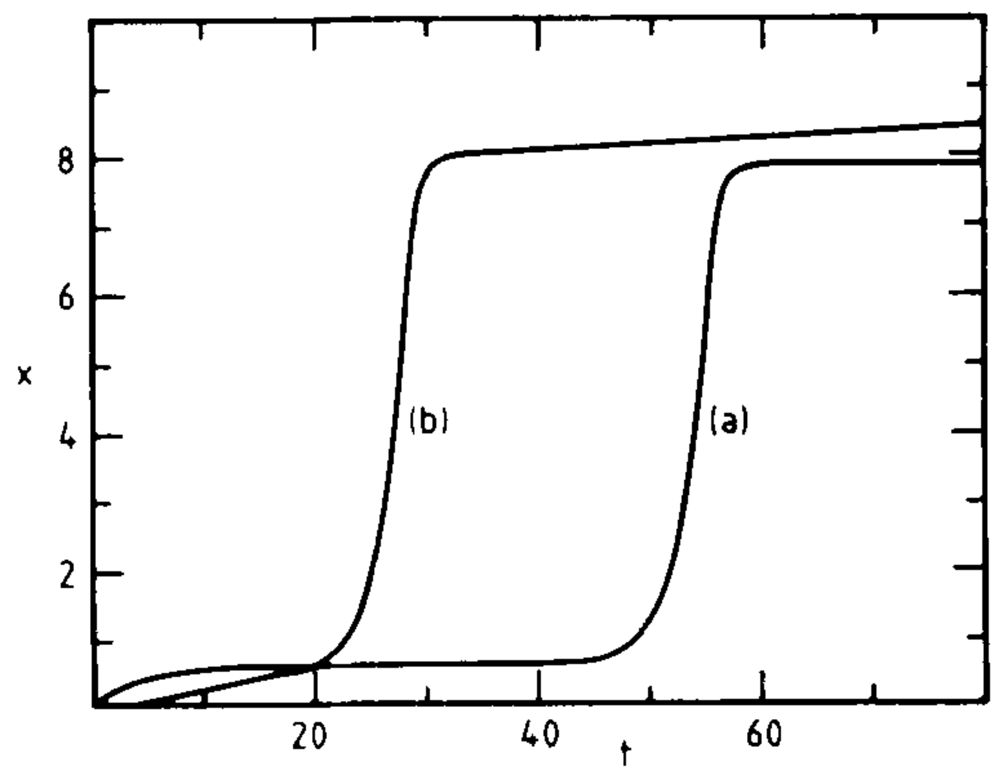
- Piecewise linear dependence of a on T (ice-albedo feedback).
- Linearized version of the Stefan-Boltzmann law for I, accounting also for the effect of increasing CO_2 concentration

$$I(T) = A + BT - K$$

Fixed K : Transitions between steady states via limit point-like bifurcations.



Time-dependent K : Stabilization of unstable states corresponding to a cool climate and slowing down prior to abrupt transitions toward a hot climate, in agreement with the predictions of the normal form analysis.



4. Conclusions

Climatic change viewed as the response of a nonlinear dynamical system to time-dependent control parameters in the presence of noise.

Sorting out some generic trends thanks to the reduction of the multivariate dynamics into a low-order one in the vicinity of transition points.

Identification of local and global forerunners of the transitions between steady states occurring through a pitchfork or a limit point bifurcation:

- Transient stabilization of unstable states tend to delay the transition.
- Fluctuations start growing at a finite distance from the transition.
- Fluctuation-driven evolutions. Frozen states.

Apply the general procedure to more realistic models than the global energy balance model.

Extend to more complex transitions and to time-dependences beyond the linear ramp.